

Optimal Control of Nonlocal Thermistor Equations

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We are concerned with the optimal control problem of the well known nonlocal thermistor problem, i.e., in studying the heat transfer in the resistor device whose electrical conductivity is strongly dependent on the temperature. Existence of an optimal control is proved. The optimality system consisting of the state system coupled with adjoint equations is derived, together with a characterization of the optimal control. Uniqueness of solution to the optimality system, and therefore the uniqueness of the optimal control, is established. The last part is devoted to numerical simulations.

Keywords: thermistor problem; partial differential equations; optimal control; existence and uniqueness; regularity; optimality system.

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1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N with a sufficiently smooth boundary $\partial\Omega$, and let $Q_T = \Omega \times (0, T)$. In this work we are interested to study an optimal control problem to the following nonlocal parabolic boundary value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \text{ in } Q_T = \Omega \times (0, T), \\ \frac{\partial u}{\partial \nu} &= -\beta u, \text{ on } S_T = \partial\Omega \times (0, T), \\ u(0) &= u_0, \text{ in } \Omega, \end{aligned} \tag{1}$$

where Δ is the Laplacian with respect to the spacial variables, f is supposed to be a smooth function prescribed below, and T a fixed positive real. Here ν denotes the outward unit normal and $\frac{\partial}{\partial \nu} = \nu \cdot \nabla$ is the normal derivative on $\partial\Omega$. Such problems arise in many applications, for instance, in studying the heat transfer in a resistor device whose electrical conductivity f is strongly dependent on the temperature u . The equation (1) describes the diffusion of the temperature with the presence of a nonlocal term. Constant λ is a dimensionless parameter, which can be identified with the square of the applied potential difference at the ends of the conductor. Function β is the positive thermal transfer coefficient, which can depend only in spatial variables

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x or time t , but for the sake of generality we take β depending in both x and t . The given value u_0 is the initial condition for temperature. Boundary conditions are derived from Newton cooling law, sometimes called Robin conditions or third type boundary conditions. In the particular case when $\beta = 0$, we obtain an homogeneous Neumann condition or an adiabatic condition. Other boundary conditions appear naturally, but for the sake of simplicity we consider in this paper mixed conditions only. Recall that under restrictive conditions, (1) is obtained by reducing the elliptic-parabolic system of partial differential equations modelling the so-called thermistor:

$$\begin{aligned} u_t &= \nabla \cdot (k(u) \nabla u) + \sigma(u) |\nabla \varphi|^2, \\ \nabla(\sigma(u) \nabla \varphi) &= 0, \end{aligned} \tag{2}$$

where u represents the temperature generated by the electric current flowing through a conductor, φ the electric potential, and $\sigma(u)$ and $k(u)$ the electric and thermal conductivities, respectively. For more description, we refer to (Lacey 1995, Tzanetis 2002). A throughout discussion about the history of thermistors, and more detailed accounts of their advantages and applications to industry, can be found in (Maclen 1979, Shi et al. 1993, Kwok 1995, Cimatti 2011). Since the paper of Rodrigues (1992), which apparently was the first who proved the existence of weak solutions to the system (2), several results were obtained. In (Antontsev and Chipot 1994) existence and regularity of weak solutions to the thermistor problem were established. We remember that existence and uniqueness of solution to (1) under hypotheses (H1)–(H3) below (cf. Sec. 2) has been established in (El Hachimi and Sidi Ammi 2005). For more on existence and uniqueness we refer to (Sidi Ammi 2010, Zhou and Liu 2010, Cimatti 2011).

Optimal control of problems governed by partial differential equations is a fertile field of research and a source of many challenging mathematical issues and interesting applications (Lions 1971, Arantes and Muñoz Rivera 2010, Tröltzsch 2010). Among essential points in the theory we mention: (i) existence, regularity, and uniqueness of the optimal control problem; (ii) necessary optimality conditions, which consist of the equation under consideration and an adjoint system. Existence and regularity theory of elliptic and parabolic equations was developed since (Ladyzenskaya et al. 1971). Optimal control theory for the system (2) received recently an important increase of interest. Results for (1) are, however, scarcer and underdeveloped. To the best of the author's knowledge, known results on the optimal control of a thermistor problem reduce to the ones of (Lee and Shilkin 2005), where the term source is taken to be the control. In (Cimatti 2007) the problem of finding the optimal difference of applied potential to the thermistor problem (2), in the sense of minimizing a suitable cost functional involving the temperature, is studied. Main result of (Cimatti 2007) gives the optimal system in the simplest case of a constant electric conductivity. In addition, a theorem of existence of the optimal solution is given in the general case of conductivities depending on the temperature. Paper (Sidi Ammi 2007) investigates a parabolic-elliptic system similar to (2), assuming a particular structure of the controls. In (Hryniv et al. 2008), authors considered the optimal control of a two dimensional steady state thermistor problem. An optimal control problem of a two dimensional time dependent thermistor system is considered in (Hryniv 2009). In (Sidi Ammi and Torres 2007) a similar problem to (2) is studied, consisting of nonlinear partial differential equations resulting from the traditional modelling of oil engineering within the framework of the mechanics of a continuous medium. The main technique of (Sidi Ammi and Torres 2007) is the adjoint state and disturbance method to derive the necessary optimality conditions. Recently, the authors in (Hömberg et al. 2009/10) investigated the state-constrained optimal control of the thermistor problem with the restriction to two-dimensional domains, while in (Cimatti 2011) some applications to the thermistor problem, and to certain problems of filtration of fluids in a porous medium in the presence of the so-called Soret–Dufour effect, are given. However, we are not aware of any work or study about the optimal control of (1).

It is known that large temperature gradients may cause a thermistor to crack. Numerical experiments in (Fowler et al. 1992, Zhou and Westbrook 1997, Nikolopoulos and Zouraris 2008)

show that low values of the heat transfer coefficient β results in small temperature variations. On the other hand, low values of the heat transfer coefficient leads to high operating temperatures of a thermistor, which is undesirable from the point of view of applications. This motivates the choice of the heat transfer coefficient as the control, and to consider the optimal control problem of minimizing the heat transfer coefficient while keeping the operating temperature of the thermistor not too high.

2 Outline of the paper and Hypotheses

We consider an optimal control problem with the partial differential equations (1):

- (i) The control β belongs to the set of admissible controls

$$U_M = \{\beta \in L^\infty(\Omega \times (0, T)), 0 < m \leq \beta \leq M\}.$$

- (ii) The goal is to minimize a cost functional $J(\beta)$ defined in terms of $u(\beta)$ and β as

$$J(\beta) = \int_{Q_T} u dx dt + \int_{S_T} \beta^2 ds dt.$$

More precisely, we intend to find $\bar{\beta} \in U_M$ such that

$$J(\bar{\beta}) = \min_{\beta \in U_M} J(\beta). \quad (3)$$

In Section 3, existence and regularity of the optimal control are established through a minimizing sequence argument. The energy estimates, in an appropriate space, and then the class of weak solutions obtained, allow us to study, in Section 4, the optimal control problem and to derive the optimality system. The obtained necessary optimality conditions consist of the original state parabolic equation (1) coupled with the adjoint equations together with a characterization of the optimal control. In general terms, the approach used here is close to the method used in (Hryniv 2009) for investigation of the time dependent thermistor problem. Since our objective functional depends on u , it is differentiated with respect to the control. We calculate the Gâteaux derivative of J with respect to β in the direction l at the minimizer control β . We also need to differentiate u with respect to the control β . The difference quotient $(u(\beta + \varepsilon l) - u(\beta)) / \varepsilon$ is proved to converge weakly in $H^1(\Omega)$ to ψ . As a result, the function ψ verifies a linear PDE which gives the adjoint system, and an explicit form of the optimal control is determined. Section 5 is devoted to the uniqueness of the solution to the optimality system, and therefore the uniqueness of the optimal control. Finally, in Section 6 we solve the optimality system numerically for a constant case of the optimization parameter.

In the sequel we shall assume the following assumptions:

- (H1) $f : \mathbb{R} \rightarrow \mathbb{R}$ is a positive Lipschitzian continuous function.
- (H2) There exist positive constants c and α such that $c \leq f(\xi) \leq c|\xi|^{\alpha+1} + c$ for all $\xi \in \mathbb{R}$.
- (H3) $u_0 \in L^\infty(\Omega)$.

We say that u is a weak solution to (1) if

$$\int_{\Omega} \frac{\partial u}{\partial t} v dx + \int_{\Omega} \nabla u \nabla v dx + \int_{\partial\Omega} \beta u v ds = \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \int_{\Omega} f(u) v dx, \quad (4)$$

for all $v \in H^1(\Omega)$. We use the standard notation for Sobolev spaces. We denote $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ for each $p \in [1, \infty]$. Along the text constants c are generic, and may change at each occurrence.

3 Existence of an optimal control

The proof of existence of an optimal control (Theorem 3.1) is done using proper estimates (Lemma 3.2).

Theorem 3.1: *Assume that the assumptions (H1)–(H3) hold. Then, there exists at least an optimal solution $\beta \in L^\infty(Q_T)$ of (3). Function $u = u(\beta)$ verifies (1), in the sense of distributions, with the following regularity: $u \in C(0, T, L^2(\Omega))$, $\frac{\partial u}{\partial t} \in L^2(0, T, H^{-1}(\Omega))$, $u \in L^2(0, T, H^1(\Omega))$.*

Proof Let $(\beta_n)_n$ be a minimizing sequence of $J(\beta)$ in U_M . In other words, we have

$$\lim_{n \rightarrow +\infty} J(\beta_n) = \inf_{\beta \in U_M} J(\beta).$$

In order to continue the proof we proceed with the derivation of a priori estimates:

Lemma 3.2: *Let $u_n = u(\beta_n)$ be the corresponding solutions to the weak formulation of (1). Then $\|u_n\|_{L^2(0, T, H^1(\Omega))}^2 + \|u_n\|_2^2 \leq c$, where c is a constant independent of n .*

Proof Multiplying the corresponding equations of (1) by u_n and using the fact that $u_n \in L^\infty(\Omega)$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|u_n\|_2^2 + \int_{\Omega} |\nabla u_n|^2 dx + \int_{\partial\Omega} \beta_n u_n^2 ds &\leq c \int_{\Omega} |f(u_n) u_n| dx \\ &\leq c \int_{\Omega} (|u_n|^{\alpha+1} + c) |u_n| dx \\ &\leq c \|u_n\|_1 \\ &\leq c \|u_n\|_2 \\ &\leq c \|u_n\|_{H^1(\Omega)}. \end{aligned}$$

Using the fact that $0 < m \leq \beta_n$, we have

$$\frac{1}{2} \frac{\partial}{\partial t} \|u_n\|_2^2 + \int_{\Omega} |\nabla u_n|^2 dx + m \int_{\partial\Omega} u_n^2 ds \leq c \|u_n\|_{H^1(\Omega)}. \quad (5)$$

Now denote

$$\|v\|_*^2 = \int_{\Omega} |\nabla v|^2 dx + m \int_{\partial\Omega} v^2 ds. \quad (6)$$

It is well known that $\|v\|_*$ defines a norm on $H^1(\Omega)$ which is equivalent to the $\|\cdot\|_{H^1(\Omega)}$ norm (Zeidler 1988). Then, there exists a constant $\mu > 0$ such that

$$\mu \|u_n\|_{H^1(\Omega)}^2 \leq \|u_n\|_*^2 \leq c \|u_n\|_{H^1(\Omega)}^2. \quad (7)$$

It follows from (5)–(7) that

$$\frac{1}{2} \frac{\partial}{\partial t} \|u_n\|_2^2 + \mu \|u_n\|_{H^1(\Omega)}^2 \leq \frac{1}{2} \frac{\partial}{\partial t} \|u_n\|_2^2 + \|u_n\|_*^2 \leq c \|u_n\|_{H^1(\Omega)}^2 \leq \frac{\mu}{2} \|u_n\|_{H^1(\Omega)}^2 + c.$$

We obtain $\|u_n(t)\|_2^2 + \mu \|u_n\|_{L^2(0, T, H^1(\Omega))}^2 \leq c$ integrating over $(0, T)$. \square

We now continue the proof of Theorem 3.1. By Lemma 3.2 we have, for all n , that

$$u_n \in L^\infty(0, T, L^2(\Omega)) \bigcap L^2(0, T, H^1(\Omega)).$$

Therefore, from (1), $\frac{\partial u_n}{\partial t}$ is bounded in $L^2(0, T, H^{-1}(\Omega))$. Using compactness arguments of Lions (Lions 1969) and Aubin's lemma, we have that (u_n) is compact in $L^2(Q_T)$. Hence we can extract from (u_n) a subsequence, not relabeled, and there exists $\beta \in U_M$ such that

$$\begin{aligned} u_n &\rightarrow u \text{ weakly in } L^2(0, T, H^1(\Omega)), \\ \frac{\partial u_n}{\partial t} &\rightarrow \frac{\partial u}{\partial t} \text{ weakly in } L^2(0, T, H^{-1}(\Omega)), \\ u_n &\rightarrow u \text{ strongly in } L^2(Q_T), \\ u_n &\rightarrow u \text{ a.e. in } L^2(Q_T), \\ \beta_n &\rightarrow \beta \text{ weakly in } L^2(\partial\Omega), \\ \beta_n &\rightarrow \beta \text{ weakly star in } L^\infty(\partial\Omega). \end{aligned} \tag{8}$$

Our task consists now to prove that $u = u(\beta)$ is a weak solution of (1) with control β . From the weak formulation of u_n we have

$$\int_{\Omega} \frac{\partial u_n}{\partial t} v dx + \int_{\Omega} \nabla u_n \nabla v dx + \int_{\partial\Omega} \beta_n u_n v ds = \frac{\lambda}{(\int_{\Omega} f(u_n) dx)^2} \int_{\Omega} f(u_n) v dx.$$

We first show that for any test function $v \in H^1(\Omega)$ and $n \rightarrow \infty$ we have

$$\int_{\partial\Omega} \beta_n u_n v ds \rightarrow \int_{\partial\Omega} \beta u v ds.$$

Indeed,

$$\begin{aligned} \left| \int_{\partial\Omega} \beta_n u_n v ds - \int_{\partial\Omega} \beta u v ds \right| &\leq \left| \int_{\partial\Omega} (\beta_n u_n v - \beta_n u v) ds \right| + \left| \int_{\partial\Omega} (\beta_n u v - \beta u v) ds \right| \\ &\leq M \int_{\partial\Omega} |u_n - u| |v| ds + \left| \int_{\partial\Omega} (\beta_n - \beta) u v ds \right| \\ &\leq M \|u_n - u\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} + \left| \int_{\partial\Omega} (\beta_n - \beta) u v ds \right| \\ &\leq M \|u_n - u\|_{H^1(\Omega)} \|v\|_{L^2(\partial\Omega)} + \left| \int_{\partial\Omega} (\beta_n - \beta) u v ds \right|, \end{aligned} \tag{9}$$

where we used here the trace inequality $\|u\|_{L^2(\partial\Omega)} \leq c \|u\|_{H^1(\Omega)}$, which gives that $u \in H^1(\Omega)$ implies $u \in L^2(\partial\Omega)$. It is obvious from limits (8) that the right hand side of the above inequality (9) goes to 0 when $n \rightarrow \infty$. On the other hand, we have $u_n \rightarrow u$ a.e. in $\Omega \times (0, T)$. Since f is continuous, $f(u_n) \rightarrow f(u)$ a.e. in $L^2(\Omega)$. It follows that

$$\int_{\Omega} f(u_n) dx \rightarrow \int_{\Omega} f(u) dx,$$

and

$$\int_{\Omega} f(u_n) v dx \rightarrow \int_{\Omega} f(u) v dx, \quad \forall v \in H^1(\Omega).$$

We conclude that $u = u(\beta)$ is a weak solution of (1). Using the fact that $J(\beta)$ is weak lower semicontinuous with respect to the L^2 norm, it follows that the infimum is achieved at β . \square

4 Characterization of the optimal control

To study the optimal control we derive an optimality system consisting of equation (1) coupled with an adjoint system. Then, in order to obtain necessary conditions for the optimality system, we differentiate the cost functional and the temperature u with respect to the control β . Here, besides (H1)–(H3), we further suppose that

(H4) f is of class C^1 .

Theorem 4.1: *Assume hypotheses (H1)–(H4). Then $\beta \mapsto u(\beta)$ is differentiable in the sense that as $\varepsilon \rightarrow 0$*

$$\frac{u(\beta + \varepsilon l) - u(\beta)}{\varepsilon} \rightarrow \psi \text{ weakly in } H^1(\Omega),$$

for any $\beta, l \in U_M$ such that $(\beta + \varepsilon l) \in U_M$ for small ε . Moreover, ψ verifies

$$\begin{aligned} \frac{\partial \psi}{\partial t} - \Delta \psi &= \frac{-2\lambda f(u)}{(\int_{\Omega} f(u) dx)^3} \int_{\Omega} f'(u) \psi dx + \frac{\lambda f'(u) \psi}{(\int_{\Omega} f(u) dx)^2} \quad \text{in } \Omega, \\ \frac{\partial \psi}{\partial \nu} + \beta \psi + l u &= 0 \quad \text{on } \partial \Omega. \end{aligned} \tag{10}$$

The proof of Theorem 4.1 passes by several steps.

4.1 A priori estimates and convergence

Denote $u = u(\beta)$ and $u_{\varepsilon} = u(\beta_{\varepsilon})$, where $\beta_{\varepsilon} = \beta + \varepsilon l$. Before the derivation of the optimality system, we need to establish an H^1 norm estimate of $\frac{u_{\varepsilon} - u}{\varepsilon}$.

Lemma 4.2: *We have*

$$\left\| \frac{u_{\varepsilon} - u}{\varepsilon} \right\|_2^2 + \left\| \frac{u_{\varepsilon} - u}{\varepsilon} \right\|_{L^2(0,T,H^1(\Omega))}^2 \leq c.$$

Proof Subtracting equation (1) from the corresponding equation of u_{ε} , we have

$$\frac{\partial}{\partial t} \left(\frac{u_{\varepsilon} - u}{\varepsilon} \right) - \Delta \left(\frac{u_{\varepsilon} - u}{\varepsilon} \right) = \frac{\lambda (f(u_{\varepsilon}) - f(u))}{\varepsilon (\int_{\Omega} f(u_{\varepsilon}) dx)^2} + \frac{\lambda}{\varepsilon} f(u) \left(\frac{1}{(\int_{\Omega} f(u_{\varepsilon}) dx)^2} - \frac{1}{(\int_{\Omega} f(u) dx)^2} \right). \tag{11}$$

Multiplying the equation (11) by $\frac{u_\varepsilon - u}{\varepsilon}$, we obtain that

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx - \int_{\partial\Omega} \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \cdot \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \nu ds \\
&= \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx - \int_{\partial\Omega} \frac{\partial}{\partial \nu} \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \left(\frac{u_\varepsilon - u}{\varepsilon} \right) ds \\
&= \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx + \int_{\partial\Omega} \beta \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 ds + \int_{\partial\Omega} l u_\varepsilon \left(\frac{u_\varepsilon - u}{\varepsilon} \right) ds \\
&\leq \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \left\langle \frac{f(u_\varepsilon) - f(u)}{\varepsilon}, \frac{u_\varepsilon - u}{\varepsilon} \right\rangle \\
&\quad + \left\langle \lambda \frac{f(u)}{\varepsilon} \left(\frac{1}{(\int_{\Omega} f(u_\varepsilon) dx)^2} - \frac{1}{(\int_{\Omega} f(u) dx)^2} \right), \frac{u_\varepsilon - u}{\varepsilon} \right\rangle.
\end{aligned}$$

Since $0 < m \leq \beta$, we get

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx - \int_{\partial\Omega} \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \nu ds \\
&\leq \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx + m \int_{\partial\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 ds + \int_{\partial\Omega} l u_\varepsilon \left(\frac{u_\varepsilon - u}{\varepsilon} \right) ds \\
&\leq \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \left\langle \frac{f(u_\varepsilon) - f(u)}{\varepsilon}, \frac{u_\varepsilon - u}{\varepsilon} \right\rangle \\
&\quad + \left\langle \lambda \frac{f(u)}{\varepsilon} \left(\frac{1}{(\int_{\Omega} f(u_\varepsilon) dx)^2} - \frac{1}{(\int_{\Omega} f(u) dx)^2} \right), \frac{u_\varepsilon - u}{\varepsilon} \right\rangle.
\end{aligned}$$

Using the fact that f is Lipschitzian, it follows from the L^∞ boundedness of u and u_ε that

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx + m \int_{\partial\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 ds + \int_{\partial\Omega} l u_\varepsilon \left(\frac{u_\varepsilon - u}{\varepsilon} \right) ds \\
&\leq c \int_{\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 dx + \frac{\lambda (\int_{\Omega} f(u) dx)^2 - (\int_{\Omega} f(u_\varepsilon) dx)^2}{(\int_{\Omega} f(u_\varepsilon) dx)^2 (\int_{\Omega} f(u) dx)^2} \left\langle f(u), \frac{u_\varepsilon - u}{\varepsilon} \right\rangle \\
&\leq c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \frac{c}{\varepsilon} \left(\int_{\Omega} (f(u) - f(u_\varepsilon)) dx \right) \left(\int_{\Omega} (f(u) + f(u_\varepsilon)) dx \right) \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_1 \\
&\leq c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_1 \int_{\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right) dx \leq c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_1^2.
\end{aligned}$$

Since $L^2(\Omega) \subseteq L^1(\Omega)$, then

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_{\Omega} \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx + m \int_{\partial\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 ds + \int_{\partial\Omega} l u_\varepsilon \left(\frac{u_\varepsilon - u}{\varepsilon} \right) ds \\
&\leq c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2.
\end{aligned}$$

Using the trace inequality $\|u_\varepsilon\|_{L^2(\partial\Omega)} \leq c\|u_\varepsilon\|_{H^1(\Omega)}$, we have

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \int_\Omega \left| \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right|^2 dx + m \int_{\partial\Omega} \left(\frac{u_\varepsilon - u}{\varepsilon} \right)^2 ds \\
& \leq \int_{\partial\Omega} |l| |u_\varepsilon| \left| \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \right| ds + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 \\
& \leq c \|u_\varepsilon\|_{L^2(\partial\Omega)} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{L^2(\partial\Omega)} + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 \\
& \leq c \|u_\varepsilon\|_{H^1(\Omega)} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)} + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2.
\end{aligned}$$

Thus,

$$\frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_*^2 \leq c \|u_\varepsilon\|_{H^1(\Omega)} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)} + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2.$$

On the other hand, by the equivalence of $\|\cdot\|_*$ and $\|\cdot\|_{H^1}$, we have for a positive constant c that

$$c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)}^2 \leq \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_*^2.$$

It follows from Young's inequality that

$$\begin{aligned}
\frac{1}{2} \frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)}^2 & \leq c \|u_\varepsilon\|_{H^1(\Omega)} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)} + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 \\
& \leq c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)} + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 \\
& \leq \frac{c}{2} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)}^2 + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + c.
\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial t} \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_2^2 + c \left\| \frac{u_\varepsilon - u}{\varepsilon} \right\|_{H^1(\Omega)}^2 \leq c.$$

We get the intended result of Lemma 4.2 integrating this inequality with respect to time. \square

Using the energy estimates of Lemma 4.2 we have, up to a subsequence of $\varepsilon \rightarrow 0$, that there

exists ψ such that

$$\begin{aligned}
\frac{u_\varepsilon - u}{\varepsilon} &\rightarrow \psi \text{ weakly in } L^\infty(0, T, L^2(\Omega)), \\
\frac{u_\varepsilon - u}{\varepsilon} &\rightarrow \psi \text{ weakly in } L^2(0, T, H^1(\Omega)), \\
\frac{\partial}{\partial t} \left(\frac{u_\varepsilon - u}{\varepsilon} \right) &\rightarrow \frac{\partial \psi}{\partial t} \text{ weakly in } L^2(0, T, H^{-1}(\Omega)), \\
\frac{u_\varepsilon - u}{\varepsilon} &\rightarrow \psi \text{ weakly in } L^\infty(0, T, L^2(\partial\Omega)), \\
\beta_\varepsilon &\rightarrow \beta \text{ weakly in } L^2(\partial\Omega) \text{ as } \varepsilon \rightarrow 0, \\
\beta_\varepsilon &\rightarrow \beta \text{ weakly in } L^\infty(\Omega) \text{ as } \varepsilon \rightarrow 0.
\end{aligned} \tag{12}$$

4.2 Proof of Theorem 4.1

We are now ready to derive system (10). We have

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial t} \left(\frac{u_\varepsilon - u}{\varepsilon} \right) v dx + \int_\Omega \nabla \left(\frac{u_\varepsilon - u}{\varepsilon} \right) \nabla v dx + \int_{\partial\Omega} \beta \left(\frac{u_\varepsilon - u}{\varepsilon} \right) v ds + \int_{\partial\Omega} l u_\varepsilon v ds \\
&= I + II
\end{aligned} \tag{13}$$

with

$$I := \frac{\lambda}{\left(\int_\Omega f(u_\varepsilon) dx \right)^2} \int_\Omega \frac{f(u_\varepsilon) - f(u)}{\varepsilon} \cdot v dx$$

and

$$II := \frac{\lambda}{\varepsilon} \left(\frac{1}{\left(\int_\Omega f(u_\varepsilon) dx \right)^2} - \frac{1}{\left(\int_\Omega f(u) dx \right)^2} \right) \int_\Omega f(u) v dx.$$

We can write II as follows:

$$\begin{aligned}
II &= \frac{\lambda}{\varepsilon} \frac{\left(\int_\Omega f(u) dx \right)^2 - \left(\int_\Omega f(u_\varepsilon) dx \right)^2}{\left(\int_\Omega f(u) dx \right)^2 \left(\int_\Omega f(u_\varepsilon) dx \right)^2} \int_\Omega f(u) v dx \\
&= \lambda \int_\Omega \frac{(f(u) - f(u_\varepsilon))}{\varepsilon} dx \times \frac{\int_\Omega (f(u) + f(u_\varepsilon)) dx}{\left(\int_\Omega f(u_\varepsilon) dx \right)^2 \left(\int_\Omega f(u) dx \right)^2} \int_\Omega f(u) v dx.
\end{aligned}$$

One can show, using weak convergence (12), that

$$II \rightarrow \frac{-2\lambda \int_\Omega f(u) v dx}{\left(\int_\Omega f(u) dx \right)^3} \int_\Omega f'(u) \psi dx \text{ as } \varepsilon \rightarrow 0.$$

In the same manner we have

$$I \rightarrow \frac{\lambda}{\left(\int_\Omega f(u) dx \right)^2} \int_\Omega f'(u) \psi v dx \text{ as } \varepsilon \rightarrow 0.$$

Again, from the weak convergence (12), we conclude that, as $\varepsilon \rightarrow 0$, (13) converges to

$$\begin{aligned} \int_{\Omega} \frac{\partial \psi}{\partial t} v dx + \int_{\Omega} \nabla \psi \nabla v + \int_{\partial \Omega} \beta \psi v ds + \int_{\partial \Omega} l u v ds \\ = \frac{-2\lambda \int_{\Omega} f(u) v dx}{(\int_{\Omega} f(u) dx)^3} \int_{\Omega} f'(u) \psi dx + \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \int_{\Omega} f'(u) \psi v dx \end{aligned}$$

for every $v \in H^1(\Omega)$. In other words,

$$\begin{aligned} \int_{\Omega} \frac{\partial \psi}{\partial t} v dx + \int_{\Omega} \nabla \psi \nabla v dx + \int_{\partial \Omega} (\beta \psi + l u) v ds \\ = \frac{-2\lambda \int_{\Omega} f(u) v dx}{(\int_{\Omega} f(u) dx)^3} \int_{\Omega} f'(u) \psi dx + \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \int_{\Omega} f'(u) \psi v dx. \quad (14) \end{aligned}$$

We can rewrite (14) as follows:

$$\begin{aligned} \int_{\Omega} \frac{\partial \psi}{\partial t} v dx + \int_{\Omega} -\Delta \psi v dx + \int_{\partial \Omega} \left(\frac{\partial \psi}{\partial \nu} + \beta \psi + l u \right) v ds \\ = \frac{-2\lambda \int_{\Omega} f(u) v dx}{(\int_{\Omega} f(u) dx)^3} \int_{\Omega} f'(u) \psi dx + \frac{\lambda}{(\int_{\Omega} f(u) dx)^2} \int_{\Omega} f'(u) \psi v dx. \end{aligned}$$

We conclude that ψ satisfies the system

$$\begin{aligned} \frac{\partial \psi}{\partial t} - \Delta \psi &= \frac{-2\lambda \int_{\Omega} f'(u) \psi dx}{(\int_{\Omega} f(u) dx)^3} f(u) + \frac{\lambda f'(u) \psi}{(\int_{\Omega} f(u) dx)^2} \quad \text{in } \Omega, \\ \frac{\partial \psi}{\partial \nu} + \beta \psi + l u &= 0 \quad \text{on } \partial \Omega. \end{aligned}$$

This completes the proof of Theorem 4.1.

4.3 Derivation of the adjoint system

In order to derive the optimality system and to characterize the optimal control, we introduce an adjoint function φ , defined in Q_T and enough smooth, and the adjoint operator associated with ψ . Multiplying the first equation of (10) by φ and integrating in space and time, we have

$$\begin{aligned} \int_{Q_T} \frac{\partial \psi}{\partial t} \cdot \varphi dx dt + \int_{Q_T} -\Delta \psi \cdot \varphi dx dt \\ = \frac{-2\lambda \int_{\Omega} f'(u) \psi dx}{(\int_{\Omega} f(u) dx)^3} \int_{Q_T} f(u) \varphi dx dt + \frac{\int_{Q_T} \lambda f'(u) \psi \varphi dx dt}{(\int_{\Omega} f(u) dx)^2} \quad \text{in } \Omega. \quad (15) \end{aligned}$$

Integrating by parts (15) with respect to time, and imposing the boundary and initial conditions

$$\frac{\partial \varphi}{\partial \nu} + \beta \varphi = 0 \quad \text{on } \partial \Omega \times (0, T), \quad \varphi(T) = 0, \quad \varphi(0) = 0,$$

we obtain

$$\begin{aligned} \int_{\Omega} \psi(T) \varphi(T) dx - \int_{\Omega} \psi(0) \varphi(0) dx + \int_{Q_T} -\frac{\partial \varphi}{\partial t} \cdot \psi dx dt + \int_{Q_T} -\Delta \varphi \cdot \psi dx dt \\ = \frac{-2\lambda \int_{Q_T} f(u) \varphi dx dt}{(\int_{\Omega} f(u) dx)^3} \int_{\Omega} f'(u) \psi dx + \frac{\lambda \int_{Q_T} f'(u) \varphi \psi dx dt}{(\int_{\Omega} f(u) dx)^2}. \end{aligned}$$

Thus, the function φ satisfies the adjoint system given by

$$\begin{aligned} -\frac{\partial \varphi}{\partial t} - \Delta \varphi &= \frac{-2\lambda \int_{\Omega} f(u) \varphi dx}{(\int_{\Omega} f(u) dx)^3} f'(u) + \frac{\lambda f'(u) \varphi}{(\int_{\Omega} f(u) dx)^2} + 1 \text{ in } Q_T, \\ \frac{\partial \varphi}{\partial \nu} + \beta \varphi &= 0 \text{ on } \partial \Omega \times (0, T), \\ \varphi(0) &= 0, \quad \varphi(T) = 0, \end{aligned} \tag{16}$$

where the 1 appears from differentiation of the integrand of $J(\beta)$ with respect to the state u .

Theorem 4.3 (Existence of solution to the adjoint system): *Given an optimal control $\beta \in U_M$ and the corresponding state u , there exists a solution φ to the adjoint system (16).*

Proof Follows by the arguments in (Sidi Ammi and Torres 2007). \square

4.4 Derivation of the optimality system

Gathering equation (1) and the adjoint system (16), we obtain the following optimality system:

$$\begin{aligned} u_t - \Delta u &= \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \\ -\frac{\partial \varphi}{\partial t} - \Delta \varphi &= \frac{-2\lambda \int_{\Omega} f(u) \varphi dx}{(\int_{\Omega} f(u) dx)^3} f'(u) + \frac{\lambda f'(u) \varphi}{(\int_{\Omega} f(u) dx)^2} + 1 \text{ in } Q_T, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 \text{ on } \partial \Omega \times (0, T), \\ \frac{\partial \varphi}{\partial \nu} + \beta \varphi &= 0 \text{ on } \partial \Omega \times (0, T), \\ \varphi(0) &= 0, \quad \varphi(T) = 0, \\ u(0) &= u_0. \end{aligned} \tag{17}$$

Remark 1: The existence of solution to the optimality system (17) follows from the existence of solution to the state system (1) and the adjoint system (16), gathered with the existence of optimal control.

We characterize the optimal control with the help of the arguments of (Hryniv 2009).

Lemma 4.4: *The optimal control β is explicitly given by*

$$\beta(x, t) = \min \left(\max \left(-\frac{u\varphi}{2}, m \right), M \right). \tag{18}$$

Proof Because the minimum of the cost functional J is achieved at β , using (10), the convergence results (12), and the second equation of the system (17), we have, for a variation $l \in U_M$ with

$\beta + \epsilon l \in U_M$ and $\epsilon > 0$ sufficiently small, that

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{J(\beta + \epsilon l) - J(\beta)}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \int_{Q_T} (u(\beta + \epsilon l) - u(\beta)) dx dt + \int_{\partial\Omega \times (0,T)} ((\beta + \epsilon l)^2 - \beta^2) ds dt \right\} \\
&\leq \lim_{\epsilon \rightarrow 0} \int_{Q_T} \frac{u(\beta + \epsilon l) - u(\beta)}{\epsilon} dx dt + 2 \int_{\partial\Omega \times (0,T)} \beta l ds dt \\
&\leq \int_{Q_T} \psi dx dt + 2 \int_{\partial\Omega \times (0,T)} \beta l ds dt = \int_{Q_T} (\psi, \varphi)(1, 0) dx dt + 2 \int_{\partial\Omega \times (0,T)} \beta l ds dt \\
&\leq \int_{\partial\Omega \times (0,T)} (2\beta l + l u \varphi) ds dt \\
&\leq \int_{\partial\Omega \times (0,T)} l (2\beta + u \varphi) ds dt.
\end{aligned}$$

Using the arguments and techniques in (Hryniv 2009) involving choices of the variation function l , we have three cases to distinguish. (i) Take the variation ℓ to have support on the set $\{x \in \partial\Omega : m < \beta(x, t) < M\}$. The variation $\ell(x, t)$ can be of any sign, therefore we obtain $2\beta + u\varphi = 0$, whence $\beta = -\frac{u\varphi}{2}$. (ii) On the set $\{(x, t) \in S_T = \partial\Omega \times (0, T) : \beta(x, t) = M\}$, the variation must satisfy $\ell(x, t) \leq 0$ and therefore we get $2\beta + u\varphi \leq 0$, implying $M = \beta(x, t) \leq -\frac{u\varphi}{2}$. (iii) On the set $\{(x, t) \in \partial\Omega \times (0, T) : \beta(x, t) = m\}$, the variation must satisfy $\ell(x, t) \geq 0$. This implies $2\beta + u\varphi \geq 0$ and hence $m = \beta(x) \geq -\frac{u\varphi}{2}$. Combining cases (i), (ii), and (iii) gives

$$\beta = \begin{cases} -\frac{u\varphi}{2} & \text{if } m < -\frac{u\varphi}{2} < M, \\ M, & \text{if } -\frac{u\varphi}{2} \geq M, \\ m & \text{if } -\frac{u\varphi}{2} \leq m. \end{cases}$$

This can be written compactly as (18). □

4.5 Particular case: a constant heat transfer coefficient

Let us consider now the case when the heat transfer coefficient β is a constant, i.e., when β is independent of x and t , and

$$J = \int_{\Omega} u dx + \beta^2. \quad (19)$$

We need to adjust the parameter $\beta \in U_M$ in such way that the new form of the functional (19) is minimized. Then, all the theory of existence of optimal control and derivation of the optimality system, that one needs to put into the proofs of the previous sections carries over to this case and are simpler. As for the characterization of optimal control, we have:

Lemma 4.5: *The optimal parameter characterization related to (19) is*

$$\beta = \min \left(\max \left(m, -\frac{1}{2} \int_{\partial\Omega} u \varphi ds \right), M \right). \quad (20)$$

Proof For the characterization of the optimal control we take into account the new expression

of the cost functional (19):

$$\begin{aligned}
0 &\leq \lim_{\epsilon \rightarrow 0} \frac{J(\beta + \epsilon l) - J(\beta)}{\epsilon} \\
&= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left\{ \int_{\Omega} u(\beta + \epsilon l) dx + (\beta + \epsilon l)^2 - \int_{\Omega} u(\beta) dx - \beta^2 \right\} \\
&= \lim_{\epsilon \rightarrow 0} \left\{ \int_{\Omega} \frac{u(\beta + \epsilon l) - u(\beta)}{\epsilon} dx + \frac{(\beta + \epsilon l)^2 - \beta^2}{\epsilon} \right\} \\
&= \int_{\Omega} \psi dx + 2\beta l \\
&= \int_{\Omega} (\psi, \varphi)(1, 0) dx + 2\beta l.
\end{aligned}$$

Multiplying the optimality system (17) with the test function ψ , integrating by parts, and using (10), we find that

$$0 \leq \lim_{\epsilon \rightarrow 0} \frac{J(\beta + \epsilon l) - J(\beta)}{\epsilon} = l \int_{\partial\Omega} u \varphi ds + 2\beta l.$$

Therefore,

$$l \left(\int_{\partial\Omega} u \varphi ds + 2\beta \right) \geq 0.$$

Repeating all the steps as those yielding to (18), we obtain that the optimal parameter β is characterized by (20). \square

5 Uniqueness of the optimal control

The uniqueness of the optimal control is mainly based on the L^∞ boundedness of u and φ . These are quite realistic assumptions since physical quantities are always bounded. It has been shown in (Sidi Ammi 2010) that $u \in L^\infty(\Omega)$. It remains to establish that φ is also essentially bounded.

Lemma 5.1: *Under hypotheses (H1)–(H4) one has $\varphi \in L^\infty(\Omega)$.*

Proof Multiplying the second equation of (17), governed by φ , by $-\varphi^{k+1}$ for some enough big integer $k > 2$, we have by the L^∞ estimate of u and Young's inequality that

$$\begin{aligned}
\frac{1}{k+2} \frac{\partial}{\partial t} \|\varphi\|_{k+2}^{k+2} + \int_{\Omega} \nabla \varphi \nabla (|\varphi|^{k+1}) dx &\leq c \|\varphi\|_{L^1(\Omega)} \|\varphi\|_{L^{k+1}(\Omega)}^{k+1} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+2} + \|\varphi\|_{L^{k+1}(\Omega)}^{k+1} \\
&\leq c \|\varphi\|_{L^{k+1}(\Omega)}^{k+2} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+2} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+1} \\
&\leq c \|\varphi\|_{L^{k+2}(\Omega)}^{k+2} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+1}.
\end{aligned}$$

Then,

$$\frac{1}{k+2} \frac{\partial}{\partial t} \|\varphi\|_{k+2}^{k+2} + (k+1) \int_{\Omega} |\varphi|^{k-2} |\nabla \varphi|^2 dx \leq c \|\varphi\|_{L^{k+2}(\Omega)}^{k+2} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+1}.$$

Taking into account that the second term of the left hand side is positive, we have

$$\frac{1}{k+2} \frac{\partial}{\partial t} \|\varphi\|_{L^{k+2}}^{k+2} \leq c \|\varphi\|_{L^{k+2}(\Omega)}^{k+2} + c \|\varphi\|_{L^{k+2}(\Omega)}^{k+1}.$$

Setting $y_k = \|\varphi\|_{L^{k+2}(\Omega)}$, it follows that

$$y_k^{k+1} \frac{\partial y_k}{\partial t} \leq c y_k^{k+2} + c y_k^{k+1}.$$

In other words,

$$\frac{\partial y_k}{\partial t} \leq c y_k + c.$$

By the Gronwall Lemma we have $y_k = \|\varphi\|_{L^{k+2}(\Omega)} \leq c$, where c are constants independent of k . Letting $k \rightarrow \infty$, we have $\|\varphi\|_{L^\infty(\Omega)} \leq c$. \square

Theorem 5.2: *If the hypotheses (H1)–(H4) hold, then the solution of the optimality system (17) is unique and, therefore, the optimal control β is unique.*

Proof Let u_1, φ_1 and u_2, φ_2 be two solutions to the optimality system (17) and β_1, β_2 be two optimal controls. Denote $w = u_1 - u_2$ and $\varphi = \varphi_2 - \varphi_1$. Upon subtracting and estimating the difference between the equations governed by u_1 and u_2 , we have

$$\frac{\partial}{\partial t} w - \Delta w = \frac{\lambda f(u_1)}{(\int_{\Omega} f(u_1) dx)^2} - \frac{\lambda f(u_2)}{(\int_{\Omega} f(u_2) dx)^2} = g(u_1, u_2)w, \quad (21)$$

where

$$g(u_1, u_2) = \left(\frac{\lambda f(u_1)}{(\int_{\Omega} f(u_1) dx)^2} - \frac{\lambda f(u_2)}{\int_{\Omega} f(u_2) dx} \right) / (u_2 - u_1).$$

By using hypotheses (H1)–(H4) and the L^∞ estimate of u_i , $i = 1, 2$, we have $g(u_1, u_2) \in L^\infty(\Omega)$. Multiplying (21) by w yields

$$\frac{1}{2} \frac{\partial \|w\|_2^2}{\partial t} + \int_{\Omega} |\nabla w|^2 dx + \int_{\partial\Omega} \beta_2 |w|^2 ds + \int_{\partial\Omega} (\beta_2 - \beta_1) u_1 w ds \leq c \|w\|_2^2.$$

Since $m \leq \beta_2$, we have

$$\frac{1}{2} \frac{\partial \|w\|_2^2}{\partial t} + \int_{\Omega} |\nabla w|^2 dx + m \int_{\partial\Omega} |w|^2 ds \leq c \|w\|_2^2 + \int_{\partial\Omega} |\beta_2 - \beta_1| |u_1| |w| ds.$$

It follows from $\beta_i = \min\left(\max\left(-\frac{u_i \varphi_i}{2}, m\right), M\right)$, $i = 1, 2$, that

$$|\beta_2 - \beta_1| \leq \frac{1}{2} |u_2 \varphi_2 - u_1 \varphi_1| \leq \frac{1}{2} (|u_2 \varphi| + |w \varphi_1|).$$

Then we get

$$\begin{aligned}
& \frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 + \int_{\Omega} |\nabla w|^2 dx + m \int_{\partial\Omega} |w|^2 ds \\
& \leq c \|w\|_2^2 + \frac{1}{2} \int_{\partial\Omega} |u_2 \varphi_2 - u_1 \varphi_1| |u_1| |w| ds \\
& \leq c \|w\|_2^2 + \frac{1}{2} \int_{\partial\Omega} (|u_2 \varphi| + |w \varphi_1|) |u_1| |w| ds \\
& \leq c \|w\|_2^2 + \frac{1}{2} \|u_1\|_{\infty} \|u_2\|_{\infty} \|\varphi\|_{L^2(\partial\Omega)} \|w\|_{L^2(\partial\Omega)} + \frac{1}{2} \|u_1\|_{\infty} \|\varphi_1\|_{\infty} \|w\|_{L^2(\partial\Omega)}^2
\end{aligned}$$

and, using Young's inequality,

$$\frac{1}{2} \frac{\partial}{\partial t} \|w\|_2^2 + \int_{\Omega} |\nabla w|^2 dx + \left(m - \frac{1}{2} \|u_1\|_{\infty} \|\varphi_1\|_{\infty} - c \right) \|w\|_{L^2(\partial\Omega)}^2 \leq c \|w\|_2^2 + c \|\varphi\|_{L^2(\partial\Omega)}^2. \quad (22)$$

On the other hand, using the adjoint system, we have

$$-\frac{\partial \varphi_2}{\partial t} - \Delta \varphi_2 = h_2(u_1, u_2, \varphi_1, \varphi_2)(\varphi_2 - \varphi_1) \quad (23)$$

and

$$-\frac{\partial \varphi_1}{\partial t} - \Delta \varphi_1 = h_1(u_1, u_2, \varphi_1, \varphi_2)(\varphi_2 - \varphi_1), \quad (24)$$

where

$$h_1(u_1, u_2, \varphi_1, \varphi_2) = \left(-\frac{2\lambda \int_{\Omega} f(u_1) \varphi_1 dx}{(\int_{\Omega} f(u_1) dx)^3} f'(u_1) + \frac{\lambda f'(u_1) \varphi_1}{(\int_{\Omega} f(u_1) dx)^2} + 1 \right) / (\varphi_2 - \varphi_1)$$

and

$$h_2(u_1, u_2, \varphi_1, \varphi_2) = \left(-\frac{2\lambda \int_{\Omega} f(u_2) \varphi_2 dx}{(\int_{\Omega} f(u_2) dx)^3} f'(u_2) + \frac{\lambda f'(u_2) \varphi_2}{(\int_{\Omega} f(u_2) dx)^2} + 1 \right) / (\varphi_2 - \varphi_1).$$

Note that $h_1(u_1, u_2, \varphi_1, \varphi_2), h_2(u_1, u_2, \varphi_1, \varphi_2) \in L^{\infty}(\Omega)$. Subtracting (23) from (24), we get

$$\frac{\partial(\varphi_2 - \varphi_1)}{\partial t} + \Delta(\varphi_2 - \varphi_1) = (h_1(u_1, u_2, \varphi_1, \varphi_2) - h_2(u_1, u_2, \varphi_1, \varphi_2))(\varphi_2 - \varphi_1).$$

Multiplying the above equation by $\varphi = \varphi_2 - \varphi_1$, using hypotheses, and the L^{∞} estimates of $u_1, u_2, \varphi_1, \varphi_2, h_1, h_2$, we get

$$\frac{1}{2} \frac{\partial}{\partial t} \|\varphi_2 - \varphi_1\|_2^2 - \left\{ \int_{\Omega} |\nabla(\varphi_2 - \varphi_1)|^2 dx - \int_{\partial\Omega} \frac{\partial}{\partial \nu} (\varphi_2 - \varphi_1)(\varphi_2 - \varphi_1) ds \right\} \leq c \|\varphi_2 - \varphi_1\|_2^2.$$

Then

$$\frac{1}{2} \frac{\partial}{\partial t} \|\varphi\|_2^2 - \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\partial\Omega} (\beta_1 \varphi_1 - \beta_2 \varphi_2)(\varphi_2 - \varphi_1) ds \leq c \|\varphi\|_2^2$$

and it follows that

$$\frac{1}{2} \frac{\partial}{\partial t} \|\varphi\|_2^2 - \int_{\Omega} |\nabla \varphi|^2 dx + \int_{\partial\Omega} \beta_2 |\varphi|^2 ds + \int_{\partial\Omega} (\beta_2 - \beta_1) \varphi_1 \varphi ds \leq c \|\varphi\|_2^2.$$

We have $\beta_i = \max(\min(m, -\frac{u_i \varphi_i}{2}), M)$. Therefore,

$$|\beta_2 - \beta_1| \leq |u_2 \varphi_2 - u_1 \varphi_1| \leq \frac{1}{2} (|\varphi_2 w| + |u_1 \varphi|).$$

Using the fact that $m \leq \beta_2$, we have

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} \|\varphi\|_2^2 + m \int_{\partial\Omega} |\varphi|^2 ds \\ & \leq \int_{\Omega} |\nabla \varphi|^2 dx + c \|\varphi\|_2^2 + \frac{1}{2} \int_{\partial\Omega} |u_2 \varphi_2 - u_1 \varphi_1| |\varphi_1 \varphi| ds \\ & \leq \int_{\Omega} |\nabla \varphi|^2 dx + c \|\varphi\|_2^2 + \frac{1}{2} \int_{\partial\Omega} (|\varphi_2 w| + |u_1 \varphi|) |\varphi_1 \varphi| ds \\ & \leq \int_{\Omega} |\nabla \varphi|^2 dx + c \|\varphi\|_2^2 + \frac{1}{2} \|\varphi_2\|_{\infty} \|\varphi_1\|_{\infty} \|w\|_{L^2(\partial\Omega)} \|\varphi\|_{L^2(\partial\Omega)} + \frac{1}{2} \|\varphi_1\|_{\infty} \|u_1\|_{\infty} \|\varphi\|_{L^2(\partial\Omega)}^2. \end{aligned}$$

Using again Young's inequality, we get

$$\frac{1}{2} \frac{\partial}{\partial t} \|\varphi\|_2^2 + \left\{ m - \frac{1}{2} \|\varphi_1\|_{\infty} \|u_1\|_{\infty} - c \right\} \|\varphi\|_{L^2(\partial\Omega)}^2 \leq \int_{\Omega} |\nabla \varphi|^2 dx + c \|\varphi\|_2^2 + c \|w\|_{L^2(\partial\Omega)}^2 \quad (25)$$

and, from Poincaré's inequality and the fact that the operator trace from $H^1(\Omega)$ to the boundary space $L^2(\partial\Omega)$ is linear and compact, we have from (22) and (25) that

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} (\|w\|_2^2 + \|\varphi\|_2^2) + \left(m - \frac{1}{2} \|u_1\|_{\infty} \|\varphi_1\|_{\infty} - c \right) \|w\|_{L^2(\partial\Omega)}^2 \\ & + \left\{ m - \frac{1}{2} \|\varphi_1\|_{\infty} \|u_1\|_{\infty} - c \right\} \|\varphi\|_{L^2(\partial\Omega)}^2 \leq \int_{\Omega} |\nabla \varphi|^2 dx + c \|w\|_2^2 + c \|\varphi\|_2^2 + c \|w\|_{L^2(\partial\Omega)}^2 + c \|\varphi\|_{L^2(\partial\Omega)}^2. \end{aligned}$$

Then,

$$\begin{aligned} & \frac{1}{2} \frac{\partial}{\partial t} (\|w\|_2^2 + \|\varphi\|_2^2) + \left(m - \frac{1}{2} \|u_1\|_{\infty} \|\varphi_1\|_{\infty} - c \right) \|w\|_{L^2(\partial\Omega)}^2 \\ & + \left\{ m - \frac{1}{2} \|\varphi_1\|_{\infty} \|u_1\|_{\infty} - c \right\} \|\varphi\|_{L^2(\partial\Omega)}^2 \leq c \|w\|_2^2 + c \|\varphi\|_2^2 \end{aligned}$$

and for m sufficiently large one has

$$\frac{\partial}{\partial t} (\|w\|_2^2 + \|\varphi\|_2^2) \leq c (\|w\|_2^2 + \|\varphi\|_2^2). \quad (26)$$

Gronwall's inequality leads to $\|w\|_2^2 + \|\varphi\|_2^2 \leq 0$. Then $u_1 = u_2$ and $\varphi_2 = \varphi_1$, which gives the uniqueness of solutions to the optimality system and therefore the uniqueness of the optimal control, since we have the existence of an optimal control and corresponding state and adjoint, which satisfy the optimality system. This completes the proof of Theorem 5.2. \square

Remark 2: The uniqueness of the optimal control can be obtained from

$$|\beta_2 - \beta_1| \leq \frac{1}{2}|u_2\varphi_2 - u_1\varphi_1| \leq \frac{1}{2}(|u_2\varphi| + |w\varphi_1|),$$

since $\varphi = w = 0$.

6 Numerical Example

We now give a numerical example for a particular problem. We use a finite element approach based on the Galerkin method to obtain approximate steady state solutions of the optimality system in the one-dimensional case. The formulation of the finite element method is based on a variational formulation of the continuous optimality system. The optimality system is discretized by finite differences. We then obtain the following one-dimensional nonlocal thermistor problem:

$$\frac{\partial u}{\partial t} - \Delta u = \frac{\lambda f(u)}{(\int_{\Omega} f(u) dx)^2}, \quad 0 < x < 1, \quad t > 0,$$

subject to the boundary and initial conditions

$$\frac{\partial u}{\partial x} = -\beta u \quad \text{on } \partial\Omega \times (0, T),$$

$$u(x, 0) = 0, \quad 0 \leq x \leq 1.$$

We divide the interval $\Omega = [0, 1]$ into N equal finite elements $0 = x_0 < x_1 < \dots < x_N = 1$. Let (x_j, x_{j+1}) be a partition of Ω and $x_{j+1} - x_j = h = \frac{1}{N}$ the step length. By S we denote a basis of the usual pyramid functions:

$$v_j = \begin{cases} \frac{1}{h}x + (1-j) & \text{on } [x_{j-1}, x_j], \\ -\frac{1}{h}x + (1+j) & \text{on } [x_j, x_{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

First, we write the problem in weak or variational form. We multiply the parabolic equation by v_j (for j fixed), integrate over $(0, 1)$, and apply Green's formula on the left-hand side, to obtain

$$\int_{\Omega} \frac{\partial u}{\partial t} v_j dx + \int_{\Omega} \nabla u \nabla v_j dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v_j ds = \frac{\lambda \int_{\Omega} f(u) v_j dx}{(\int_{\Omega} f(u) dx)^2}.$$

Using the boundary condition we get

$$\int_{\Omega} \frac{\partial u}{\partial t} v_j dx + \int_{\Omega} \nabla u \nabla v_j dx + \int_{\partial\Omega} \beta u v_j ds = \frac{\lambda \int_{\Omega} f(u) v_j dx}{(\int_{\Omega} f(u) dx)^2}. \quad (27)$$

We now turn our attention to the solution of system (27) by discretization with respect to the time variable. We introduce a time step τ and time levels $t_n = n\tau$, where n is a nonnegative integer, and denote by u^n the approximation of $u(t_n)$ to be determined. We use the backward Euler–Galerkin method, which is defined by replacing the time derivative in (27) by a backward

difference $\frac{u^{n+1}-u^n}{\tau}$. So the approximations u^{n+1} admit a unique representation,

$$u^{n+1} = \sum_{i=-1}^N \alpha_i^{n+1} v_i,$$

where α_i^{n+1} are unknown real coefficients to be determined. Thus,

$$\int_{\Omega} \frac{u^{n+1} - u^n}{\tau} v_j dx + \int_{\Omega} \nabla u^{n+1} \nabla v_j dx + \int_{\partial\Omega} \beta u^{n+1} v_j ds = \frac{\lambda \int_{\Omega} f(u^n) v_j dx}{(\int_{\Omega} f(u^n) dx)^2}.$$

The scheme may be stated in terms of the functions v_i : find the coefficients α_i^{n+1} in $u^{n+1} = \sum_{i=-1}^N \alpha_i^{n+1} v_i$ such that

$$\begin{aligned} \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} v_i v_j dx + \tau \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} \nabla v_i \nabla v_j dx + \tau \int_{\partial\Omega} \beta u^{n+1} v_j ds \\ = \sum_{i=-1}^N \alpha_i^n \int_{\Omega} v_i v_j dx + \tau \frac{\lambda \int_{\Omega} f(u^n) v_j dx}{(\int_{\Omega} f(u^n) dx)^2}. \end{aligned} \quad (28)$$

In matrix notation, this may be expressed as $(A + \tau B) \alpha^{n+1} = g^n = g(n\tau)$, where

$$A = (a_{ij}) \text{ with element } a_{ij} = \int_{\Omega} v_i v_j dx,$$

$$B = (b_{ij}) \text{ with } b_{ij} = \int_{\Omega} \nabla v_i \nabla v_j dx,$$

and α^{n+1} is the vector of unknowns $(\alpha_i^{n+1})_{i=-1}^N$. Since the matrix A and B are Gram matrices, in particular they are positive definite and invertible. Thus, the above system of ordinary differential equations has obviously a unique solution. We solve the system (28) for each time level. Estimating each term of (28) separately, we have:

$$\begin{aligned} \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} v_i v_j dx \\ = \sum_{i=-1}^N \alpha_i^{n+1} \int_0^1 v_i v_j dx \\ = \alpha_{j-1}^{n+1} \int_{x_{j-1}}^{x_j} v_{j-1} v_j dx + \alpha_j^{n+1} \int_{x_{j-1}}^{x_{j+1}} v_j^2 dx + \alpha_{j+1}^{n+1} \int_{x_j}^{x_{j+1}} v_j v_{j+1} dx \\ = \alpha_{j-1}^{n+1} \int_{x_{j-1}}^{x_j} v_{j-1} v_j dx + \alpha_j^{n+1} \left(\int_{x_{j-1}}^{x_j} v_j^2 dx + \int_{x_j}^{x_{j+1}} v_j^2 dx \right) + \alpha_{j+1}^{n+1} \int_{x_j}^{x_{j+1}} v_j v_{j+1} dx. \end{aligned}$$

Using the expression of v_{j-1}, v_j and v_{j+1} , we obtain

$$\sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} v_i v_j dx = \frac{h}{6} \alpha_{j-1}^{n+1} + \frac{2h}{3} \alpha_j^{n+1} + \frac{h}{6} \alpha_{j+1}^{n+1}. \quad (29)$$

Similarly, we have

$$\begin{aligned} & \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} \nabla v_i \nabla v_j dx \\ &= \sum_{i=-1}^N \alpha_i^{n+1} \int_{\Omega} \frac{\partial v_i}{\partial x} \frac{\partial v_j}{\partial x} dx \\ &= \alpha_{j-1}^{n+1} \int_{x_{j-1}}^{x_j} \frac{\partial v_{j-1}}{\partial x} \frac{\partial v_j}{\partial x} dx + \alpha_j^{n+1} \int_{x_{j-1}}^{x_{j+1}} \left(\frac{\partial v_j}{\partial x} \right)^2 dx + \alpha_{j+1}^{n+1} \int_{x_j}^{x_{j+1}} \frac{\partial v_j}{\partial x} \frac{\partial v_{j+1}}{\partial x} dx, \\ &= -\frac{\alpha_{j-1}^{n+1}}{h^2} \int_{x_{j-1}}^{x_j} dx + \frac{\alpha_j^{n+1}}{h^2} \int_{x_{j-1}}^{x_{j+1}} dx - \frac{\alpha_{j+1}^{n+1}}{h^2} \int_{x_j}^{x_{j+1}} dx \\ &= -\frac{1}{h} \alpha_{j-1}^{n+1} + \frac{2}{h} \alpha_j^{n+1} - \frac{1}{h} \alpha_{j+1}^{n+1}. \end{aligned} \quad (30)$$

On the other hand,

$$\int_{\Omega} u^n v_j = \sum_{i=-1}^N \alpha_i^n \int_{\Omega} v_i v_j dx = \frac{h}{6} \alpha_{j-1}^n + \frac{2h}{3} \alpha_j^n + \frac{h}{6} \alpha_{j+1}^n \quad (31)$$

and

$$\begin{aligned} \beta \int_{\partial\Omega=\{0,1\}} u^{n+1} v_j &\simeq \frac{1}{2} (\beta u^{n+1}(1) v_j(1) + \beta u^{n+1}(0) v_j(0)) \\ &= \frac{1}{2} (\beta \alpha_N^{n+1} v_j(1) + \beta \alpha_0^{n+1} v_j(0)) \\ &= \begin{cases} \frac{1}{2} \beta \alpha_0^{n+1} & \text{if } j = 0, \\ 0 & \text{if } j = 1 \dots N-2, \\ 0 & \text{if } j = N-1. \end{cases} \end{aligned} \quad (32)$$

Furthermore,

$$\frac{\lambda \int_{\Omega} f(u^n) v_j dx}{\left(\int_{\Omega} f(u^n) dx \right)^2} \simeq \begin{cases} \frac{2\lambda f(\alpha_0^n)}{(f(\alpha_0^n) + f(\alpha_N^n))^2} & \text{if } j = 0, \\ 0 & \text{if } j = 1, \dots, N-2, \\ 0 & \text{if } j = N-1. \end{cases} \quad (33)$$

Using the boundary conditions, we have

$$\begin{aligned}\alpha_{-1}^{n+1} &= \alpha_1^{n+1} + (h\beta + 1)\alpha_0^{n+1}, \\ \alpha_{-1}^n &= \alpha_1^n + (h\beta + 1)\alpha_0^n, \\ \alpha_N^{n+1} &= \frac{1}{\beta h + 1}\alpha_{N-1}^{n+1}, \\ \alpha_N^n &= \frac{1}{\beta h + 1}\alpha_{N-1}^n.\end{aligned}$$

From the initial condition we get $\alpha_0^0 = \alpha_N^0 = 0$. Setting

$$a = \frac{h}{6} - \frac{\tau}{h}, \quad b = \frac{2h}{3} + \frac{2\tau}{h},$$

and using together (28)–(33), we then get the following system of $N-1$ linear algebraic equations: for $j = 0$,

$$\left(a(1 + h\beta) + b + \frac{\tau\beta}{2}\right)\alpha_0^{n+1} + 2a\alpha_1^{n+1} = \frac{h}{6}(5 + h\beta)\alpha_0^n + \frac{h}{3}\alpha_1^n + \frac{2\lambda\tau f(\alpha_0^n)}{(f(\alpha_0^n) + f(\alpha_N^n))^2},$$

for $j = 1, \dots, N-2$,

$$a\alpha_{j-1}^{n+1} + b\alpha_j^{n+1} + a\alpha_{j+1}^{n+1} = \frac{h}{6}\alpha_{j-1}^n + \frac{2h}{3}\alpha_j^n + \frac{h}{6}\alpha_{j+1}^n,$$

for $j = N-1$,

$$a\alpha_{N-2}^{n+1} + \left(b + \frac{a}{1 + \beta h}\right)\alpha_{N-1}^{n+1} = \frac{h}{6}\alpha_{N-2}^n + \frac{2h}{3}\left(1 + \frac{1}{4(1 + \beta h)}\right)\alpha_{N-1}^n.$$

Similarly,

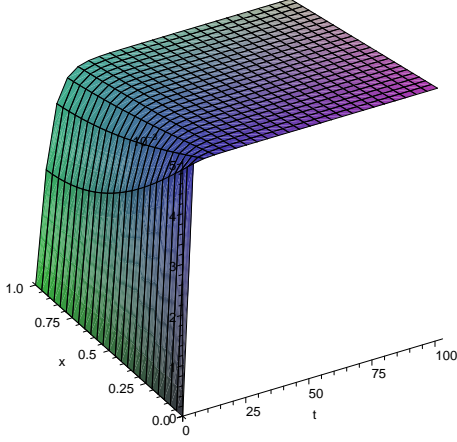
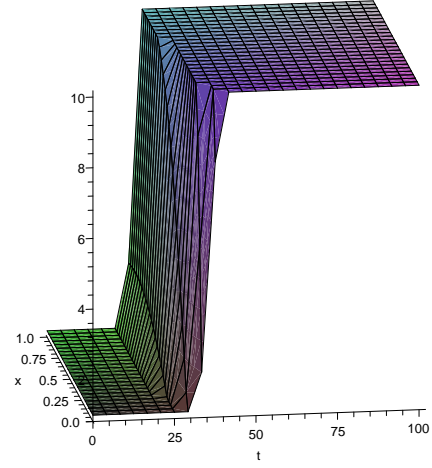
$$\varphi^{n+1} = \sum_{i=-1}^N \mu_i^{n+1} v_i,$$

where μ_i^{n+1} are unknown real coefficients to be determined. The discretization of the boundary conditions with respect to φ looks as follows:

$$\begin{aligned}\mu_{-1}^{n+1} &= \mu_1^{n+1} + (h\beta + 1)\mu_0^{n+1}, \\ \mu_{-1}^n &= \mu_1^n + (h\beta + 1)\mu_0^n, \\ \mu_N^{n+1} &= \frac{1}{1 + \beta h}\mu_{N-1}^{n+1}, \\ \mu_N^n &= \frac{1}{1 + \beta h}\mu_{N-1}^n.\end{aligned}$$

If we set

$$c = -\frac{h}{6} - \frac{\tau}{h}, \quad d = -\frac{2h}{3} + \frac{2\tau}{h},$$

Figure 1. The evolution of temperature u Figure 2. The control β

then the remaining discrete equations, approximating the optimality system, are as follows: for $j = 0$,

$$\begin{aligned} \left(c(1 + h\beta) + d + \frac{\tau\beta}{2} - \frac{2\lambda\tau\beta f'(\alpha_0^n)}{(f(\alpha_N^n) + f(\alpha_0^n))^2} \right) \mu_0^{n+1} + 2c\mu_1^{n+1} \\ = -\frac{h}{6}(5 + h\beta)\mu_0^n - \frac{h}{3}\mu_1^n + \tau h + \frac{2\lambda\tau(\mu_0^1 + \mu_1^N)f(\alpha_0^n)}{(f(\alpha_N^n) + f(\alpha_0^n))^3}, \end{aligned}$$

for $j = 1, \dots, N-2$,

$$c\mu_{j-1}^{n+1} + d\mu_j^{n+1} + c\mu_{j+1}^{n+1} = \tau h - \frac{h}{6}\mu_{j-1}^n - \frac{2h}{3}\mu_j^n - \frac{h}{6}\mu_{j+1}^n,$$

for $j = N-1$,

$$c\mu_{N-2}^{n+1} + \left(d + \frac{c}{1 + \beta h} \right) \mu_{N-1}^{n+1} = \tau h - \frac{h}{6}\mu_{N-2}^n - \frac{2h}{3} \left(1 + \frac{1}{4(1 + \beta h)} \right) \mu_{N-1}^n.$$

Finally, we have the discretization of β as follows:

$$\beta^{n+1} = \min \left(\max \left(m, -\frac{u^{n+1}\varphi^{n+1}}{2} \right), M \right). \quad (34)$$

The numerical experiments are in agreement with the results of (Sidi Ammi and Torres 2008): we obtain stable steady-state (see Figure 1). With an initial guess for the value of the control, the consecutive values of β converge to the lower bound when time is small and to the upper bound when t is big (see Figure 2).

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